A characterization of well-founded algebraic lattices

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Abstract

We characterize well-founded algebraic lattices by means of forbidden subsemilattices of the join-semilattice made of their compact elements. More specifically, we show that an algebraic lattice L is well-founded if and only if $K(L)$, the join-semilattice of compact elements of L, is well-founded and contains neither $[\omega]^{<\omega}$, nor $\underline{\Omega}(\omega^*)$ as a join-subsemilattice. As an immediate corollary, we get that an algebraic modular lattice L is well-founded if and only if $K(L)$ is well-founded and contains no infinite independent set. If $K(L)$ is a join-subsemilattice of $I_{<\omega}(Q)$, the set of finitely generated initial segments of a well-founded poset Q, then L is well-founded if and only if $K(L)$ is well-quasi-ordered.

1 Introduction and synopsis of results

Algebraic lattices and join-semilattices (with a 0) are two aspects of the same thing, as expressed in the following basic result.

Theorem 1.1 *[12], [10] The collection $J(P)$ of ideals of a join-semilattice P, once ordered by inclusion, is an algebraic lattice and the subposet $K(J(P))$ of its compact elements is isomorphic to P. Conversely, the subposet $K(L)$ of compact elements of an algebraic lattice L is a join-semilattice with a 0 and $J(K(L))$ is isomorphic to L.*

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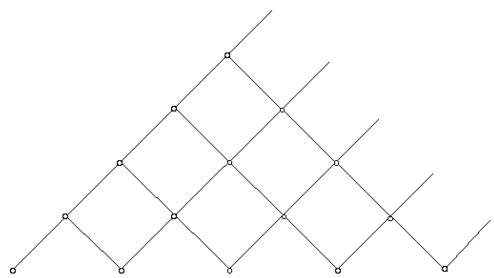

Figure 1: $\Omega(\omega^*)$

In this paper, we characterize well-founded algebraic lattices by means of forbidden join-subsemilattices of the join-semilattice made of their compact elements. In the sequel ω denotes the chain of non-negative integers, and when this causes no confusion, the first infinite cardinal as well as the first infinite ordinal . We denote ω^* the chain of negative integers. We recall that a poset P is *well-founded* provided that every non-empty subset of P has a minimal element. With the Axiom of dependent choices, this amounts to the fact that P contains no subset isomorphic to ω^*. Let $\Omega(\omega^*)$ be the set $[\omega]^2$ of two-element subsets of ω, identified to pairs (i, j), $i < j < \omega$, ordered so that $(i, j) \leq (i', j')$ if and only if $i' \leq i$ and $j \leq j'$ w.r.t. the natural order on ω. Let $\underline{\Omega}(\omega^*) := \Omega(\omega^*) \cup \{\emptyset\}$ be obtained by adding a least element. Note that $\underline{\Omega}(\omega^*)$ is isomorphic to the set of bounded intervals of ω (or ω^*) ordered by inclusion. Moreover $\underline{\Omega}(\omega^*)$ is a join-semilattice $((i, j) \vee (i', j') = (i \wedge i', j \vee j'))$. The join-semilattice $\underline{\Omega}(\omega^*)$ embeds in $\Omega(\omega^*)$ as a join-semilattice; the advantage of $\underline{\Omega}(\omega^*)$ w.r.t. our discussion is to have a zero. Let κ be a cardinal number, e.g. $\kappa := \omega$; denote $[\kappa]^{<\omega}$ (resp. $\mathfrak{P}(\kappa)$) the set, ordered by inclusion, consisting of finite (resp. arbitrary) subsets of κ. The posets $\underline{\Omega}(\omega^*)$ and $[\kappa]^{<\omega}$ are well-founded lattices, whereas the algebraic lattices $J(\underline{\Omega}(\omega^*))$ and $J([\kappa]^{<\omega})$ (κ infinite) are not well-founded (and we may note that $J([\kappa]^{<\omega})$ is isomorphic to $\mathfrak{P}(\kappa)$). As a poset $\underline{\Omega}(\omega^*)$ is isomorphic to a subset of $[\omega]^{<\omega}$, but not as a join-subsemilattice. This is our first result.

Proposition 1.2 $\underline{\Omega}(\omega^*)$ *does not embed in* $[\omega]^{<\omega}$ *as a join-subsemilattice; more generally, if* Q *is a well-founded poset then* $\underline{\Omega}(\omega^*)$ *does not embed as a join-subsemilattice into* $I_{<\omega}(Q)$*, the join-semilattice made of finitely generated initial segments of* Q*.*

Our next result expresses that $\underline{\Omega}(\omega^*)$ and $[\omega]^{<\omega}$ are unavoidable examples of well-founded join-semilattices whose set of ideals is not well-founded.

Theorem 1.3 *An algebraic lattice L is well-founded if and only if $K(L)$ is well-founded and contains no join-subsemilattice isomorphic to $\underline{\Omega}(\omega^*)$ or to $[\omega]^{<\omega}$.*

The fact that a join-semilattice P contains a join-subsemilattice isomorphic to $[\omega]^{<\omega}$ amounts to the existence of an infinite independent set. Let us recall that a subset X of a join-semilattice P is *independent* if $x \not\leq \bigvee F$ for every $x \in X$ and every non-empty finite subset F of $X \setminus \{x\}$. Conditions which may insure the existence of an infinite independent set or consequences of the inexistence of such sets have been considered within the framework of the structure of closure systems (cf. the research on the "free-subset problem" of Hajnal [21] or on the cofinality of posets [9, 16]). A basic result is the following.

Theorem 1.4 *[4] [15] Let κ be a cardinal number; for a join-semilattice P the following properties are equivalent:*
(i) P contains an independent set of size κ;
(ii) P contains a join-subsemilattice isomorphic to $[\kappa]^{<\omega}$;
(iii) P contains a subposet isomorphic to $[\kappa]^{<\omega}$;
(iv) $J(P)$ contains a subposet isomorphic to $\mathfrak{P}(\kappa)$;
(v) $\mathfrak{P}(\kappa)$ embeds in $J(P)$ via a map preserving arbitrary joins.

Let $L(\alpha) := 1 + (1 \oplus J(\alpha)) + 1$ be the lattice made of the direct sum of the one-element chain 1 and the chain $J(\alpha)$, (α finite or equal to ω^*), with top and bottom added.

Clearly $J(\underline{\Omega}(\omega^*))$ contains a sublattice isomorphic to $L(\omega^*)$. Since a modular lattice contains no sublattice isomorphic to $L(2)$, we get as a corollary of Theorem 1.3:

Theorem 1.5 *An algebraic modular lattice L is well-founded if and only if $K(L)$ is well-founded and contains no infinite independent set.*

Another consequence is this:

Theorem 1.6 *For a join-semilattice P, the following properties are equivalent:*

(i) P is well-founded with no infinite antichain ;

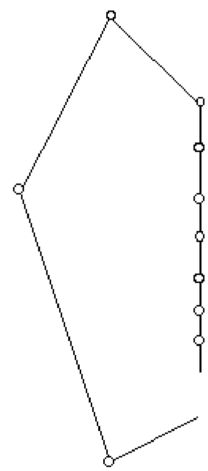

Figure 2: $L(\omega^*)$

(ii) P contains no infinite independent set and embeds as a join-semilattice into a join-semilattice of the form $I_{<\omega}(Q)$ where Q is some well-founded poset.

Posets which are well-founded and have no infinite antichain are said *well-partially-ordered* or *well-quasi-ordered*, wqo for short. They play an important role in several areas (see [8]). If P is a wqo join-semilattice then $J(P)$, the poset of ideals of P, is well-founded and one may assign to every $J \in J(P)$ an ordinal, its *height* , denoted by $h(J, J(P))$. This ordinal is defined by induction, setting $h(J, J(P)) := Sup(\{h(J', J(P)) + 1 : J' \in J(P), J' \subset J\})$ and $h(J', J(P)) := 0$ if J' is minimal in $J(P)$. The ordinal $h(J(P)) := h(P, J(P)) + 1$ is the *height* of $J(P)$. If $P := I_{<\omega}(Q)$, with Q wqo, then $J(P)$ contains a chain of order type $h(J(P))$. This is an equivalent form of the famous result of de Jongh and Parikh [6] asserting that among the linear extensions of a wqo, one has a maximum order type.

Problem 1.7 *Let P be a wqo join-semilattice; does $J(P)$ contain a chain of order type $h(J(P))$?*

An immediate corollary of Theorem 1.6 is:

Corollary 1.8 *A join-semilattice P of $[\omega]^{<\omega}$ contains either $[\omega]^{<\omega}$ as a join-semilattice or is wqo.*

Let us compare join-subsemilattices of $[\omega]^{<\omega}$. Set $P \leq P'$ for two such join-subsemilattices if P embeds in P' as a join-semilattice. This gives a quasi-order and, according to Corollary 1.8, the poset corresponding to this quasi-order has a largest element (namely $[\omega]^{<\omega}$), and all other members come from wqo join-semilattices. Basic examples of join-subsemilattices of $[\omega]^{<\omega}$ are the $I_{<\omega}(Q)$'s where Q is a countable poset such that no element is above infinitely many elements. These posets Q are exactly those which are embeddable in the poset $[\omega]^{<\omega}$ ordered by inclusion. An interesting subclass is made of posets of the form $Q = (\mathbb{N}, \leq)$ where the order $\leq$ is the intersection of the natural order $\mathfrak{N}$ on $\mathbb{N}$ and of a linear order $\mathfrak{L}$ on $\mathbb{N}$, (that is $x \leq y$ if $x \leq y$ w.r.t. $\mathfrak{N}$ and $x \leq y$ w.r.t. $\mathfrak{L}$). If α is the type of the linear order, a poset of this form is a *sierpinskisation* of α. The corresponding join-semilattices are wqo provided that the posets Q have no infinite antichain; in the particular case of a sierpinskisation of α this amounts to the fact that α is well-ordered .

As shown in [20], sierpinskisations given by a bijective map $\psi : \omega\alpha \to \omega$ which is order-preserving on each component $\omega \cdot \{i\}$ of $\omega\alpha$ are all embeddable in each other, and for this reason denoted by the same symbol $\Omega(\alpha)$. Among the representatives of $\Omega(\alpha)$, some are join-semilattices, and among them, join-subsemilattices of the direct product $\omega \times \alpha$ (this is notably the case of the poset $\Omega(\omega^*)$ we previously defined). We extend the first part of Proposition 1.2, showing that except for $\alpha \leq \omega$, the representatives of $\Omega(\alpha)$ which are join-semilattices never embed in $[\omega]^{<\omega}$ as join-semilattices, whereas they embed as posets (see Corollary 4.11 and Example 4.12). From this result, it follows that the posets $\Omega(\alpha)$ and $I_{<\omega}(\Omega(\alpha))$ do not embed in each other as join-semilattices.

These two posets provide examples of a join-semilattice P such that P contains no chain of type α while $J(P)$ contains a chain of type $J(\alpha)$. However, if α is not well ordered then $I_{<\omega}(\Omega(\alpha))$ and $[\omega]^{<\omega}$ embed in each other as join-semilattices.

Problem 1.9 *Let α be a countable ordinal. Is there a minimum member among the join-subsemilattices P of $[\omega]^{<\omega}$ such that $J(P)$ contains a chain of type $\alpha+1$? Is it true that this minimum is $I_{<\omega}(\Omega(\alpha))$ if α is indecomposable?*

2 Definitions and basic results

Our definitions and notations are standard and agree with [10] except on minor points that we will mention. We adopt the same terminology as in [4]. We recall only few things. Let P be a poset. A subset I of P is an

initial segment of P if $x \in P$, $y \in I$ and $x \leq y$ imply $x \in I$. If A is a subset of P, then $\downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$ denotes the least initial segment containing A. If $I = \downarrow A$ we say that I is *generated* by A or A is *cofinal* in I. If $A = \{a\}$ then I is a *principal initial segment* and we write $\downarrow a$ instead of $\downarrow \{a\}$. We denote $down(P)$ the set of principal initial segments of P. A *final segment* of P is any initial segment of P^*, the dual of P. We denote by $\uparrow A$ the final segment generated by A. If $A = \{a\}$ we write $\uparrow a$ instead of $\uparrow \{a\}$. A subset I of P is *directed* if every finite subset of I has an upper bound in I (that is I is non-empty and every pair of elements of I has an upper bound). An *ideal* is a non-empty directed initial segment of P (in some other texts, the empty set is an ideal). We denote $I(P)$ (respectively, $I_{<\omega}(P)$, $J(P)$) the set of initial segments (respectively, finitely generated initial segments, ideals of P) ordered by inclusion and we set $J_*(P) := J(P) \cup \{\emptyset\}$, $I_0(P) := I_{<\omega}(P) \setminus \{\emptyset\}$. Others authors use *down set* for initial segment. Note that $down(P)$ has not to be confused with $I(P)$. If P is a join-semilattice with a 0, an element $x \in P$ is *join-irreducible* if it is distinct from 0, and if $x = a \vee b$ implies $x = a$ or $x = b$ (this is a slight variation from [10]). We denote $\mathbb{J}_{irr}(P)$ the set of join-irreducibles of P. An element a in a lattice L is *compact* if for every $A \subset L$, $a \leq \bigvee A$ implies $a \leq \bigvee A'$ for some finite subset A' of A. The lattice L is *compactly generated* if every element is a supremum of compact elements. A lattice is *algebraic* if it is complete and compactly generated.

We note that $I_{<\omega}(P)$ is the set of compact elements of $I(P)$, hence $J(I_{<\omega}(P)) \cong I(P)$. Moreover $I_{<\omega}(P)$ is a lattice, and in fact a distributive lattice, if and only if P is $\downarrow$*-closed* , that is, the intersection of two principal initial segments of P is a finite union, possibly empty, of principal initial segments. We also note that $J(P)$ is the set of join-irreducible elements of $I(P)$; moreover, $I_{<\omega}(J(P)) \cong I(P)$ whenever P has no infinite antichain. Notably for the proof of Theorem 4.13, we will need the following results.

Theorem 2.1 *Let P be a poset.*
a) $I_{<\omega}(P)$ is well-founded if and only if P is well-founded (Birkhoff 1937, see [1]);
b) $I_{<\omega}(P)$ is wqo iff P is wqo iff $I(P)$ well-founded (Higman 1952 [11]);
c) if P is a well-founded join-semilattice with a 0, then every member of P is a finite join of join-irreducible elements of P (Birkhoff, 1937, see [1]);
d) A join-semilattice P with a zero is wqo if and only if every member of P is a finite join of join-irreducible elements of P and the set $\mathbb{J}_{irr}(P)$ of these join-irreducible elements is wqo (follows from b*) and* c*)).*

A poset P is *scattered* if it does not contain a copy of η, the chain of rational numbers. A topological space T is *scattered* if every non-empty closed set contains some isolated point. The power set of a set, once equipped with the product topology, is a compact space. The set $J(P)$ of ideals of a join-semilattice P with a 0 is a closed subspace of $\mathfrak{P}(P)$, hence is a compact space too. Consequently, an algebraic lattice L can be viewed as a poset and a topological space as well. It is easy to see that if L is topologically scattered then it is order scattered . It is a more significant fact, due to M.Mislove [17], that the converse holds if L is distributive.

3 Separating chains of ideals and proofs of Proposition 1.2 and Theorem 1.3

Let P be a join-semilattice. If $x \in P$ and $J \in J(P)$, then $\downarrow x$ and J have a join $\downarrow x \bigvee J$ in $J(P)$ and $\downarrow x \bigvee J = \downarrow \{x \vee y : y \in J\}$. Instead of $\downarrow x \bigvee J$ we also use the notation $\{x\} \bigvee J$. Note that $\{x\} \bigvee J$ is the least member of $J(P)$ containing $\{x\} \cup J$. We say that a non-empty chain $\mathcal{I}$ of ideals of P is *separating* if for every $I \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$ and every $x \in \cup \mathcal{I} \setminus I$, there is some $J \in \mathcal{I}$ such that $I \not\subseteq \{x\} \bigvee J$.
If $\mathcal{I}$ is separating then $\mathcal{I}$ has a least element implies it is a singleton set. In $P := [\omega]^{<\omega}$, the chain $\mathcal{I} := \{I_n : n < \omega\}$ where I_n consists of the finite subsets of $\{m : n \leq m\}$ is separating. In $P := \omega^*$, the chain $\mathcal{I} := \{\downarrow x : x \in P\}$ is non-separating, as well as all of its infinite subchains. In $P := \Omega(\omega^*)$ the chain $\mathcal{I} := \{I_n : n < \omega\}$ where $I_n := \{(i, j) : n \leq i < j < \omega\}$ has the same property.
We may observe that *a join-preserving embedding from a join-semilattice P into a join-semilattice Q transforms every separating (resp. non-separating) chain of ideals of P into a separating (resp. non-separating) chain of ideals of Q* (If $\mathcal{I}$ is a separating chain of ideals of P, then $\mathcal{J} = \{f(I) : I \in \mathcal{I}\}$ is a separating chain of ideals of Q). Hence the containment of $[\omega]^{<\omega}$ (resp. of ω^* or of $\Omega(\omega^*)$), as a join-subsemilattice, provides a chain of ideals which is separating (resp. non-separating, as are all its infinite subchains, as well). We show in the next two lemmas that the converse holds.

Lemma 3.1 *A join-semilattice P contains an infinite independent set if and only if it contains an infinite separating chain of ideals.*

Proof. Let $X = \{x_n : n < \omega\}$ be an infinite independent set. Let I_n be the ideal generated by $X \setminus \{x_i : 0 \leq i \leq n\}$. The chain $\mathcal{I} = \{I_n : n < \omega\}$ is separating. Let $\mathcal{I}$ be an infinite separating chain of ideals. Define inductively an

infinite sequence $x_0, I_0, \ldots, x_n, I_n, \ldots$ such that $I_0 \in \mathcal{I} \setminus \{\cup\mathcal{I}\}, x_0 \in \cup\mathcal{I} \setminus I_0$ and such that:
$a_n)$ $I_n \in \mathcal{I}$;
$b_n)$ $I_n \subset I_{n-1}$;
$c_n)$ $x_n \in I_{n-1} \setminus (\{x_0 \vee \ldots \vee x_{n-1}\} \bigvee I_n)$ for every $n \geq 1$.
The construction is immediate. Indeed, since $\mathcal{I}$ is infinite then $\mathcal{I} \setminus \{\cup\mathcal{I}\} \neq \emptyset$. Choose arbitrary $I_0 \in \mathcal{I} \setminus \{\cup\mathcal{I}\}$ and $x_0 \in \cup\mathcal{I} \setminus I_0$. Let $n \geq 1$. Suppose x_k, I_k defined and satisfying $a_k), b_k), c_k)$ for all $k \leq n-1$. Set $I := I_{n-1}$ and $x := x_0 \vee \ldots \vee x_{n-1}$. Since $I \in \mathcal{I}$ and $x \in \cup\mathcal{I} \setminus I$, there is some $J \in \mathcal{I}$ such that $I \not\subseteq \{x\} \bigvee J$. Let $z \in I \setminus (\{x\} \bigvee J)$. Set $x_n := z$, $I_n := J$. The set $X := \{x_n : n < \omega\}$ is independent. Indeed if $x \in X$ then since $x = x_n$ for some n, $n < \omega$, condition $c_n)$ asserts that there is some ideal containing $X \setminus \{x\}$ and excluding x. □

Lemma 3.2 *A join-semilattice P contains either ω^* or $\Omega(\omega^*)$ as a join-subsemilattice if and only if it contains an ω^*-chain $\mathcal{I}$ of ideals such that all infinite subchains are non-separating.*

Proof. Let $\mathcal{I}$ be an ω^*-chain of ideals and let A be its largest element (that is $A = \cup\mathcal{I}$). Let E denote the set $\{x : x \in A$ and $I \subset\downarrow x$ for some $I \in \mathcal{I}\}$.
Case (i). For every $I \in \mathcal{I}$, $I \cap E \neq \emptyset$. We can build an infinite strictly decreasing sequence $x_0, \ldots, x_n, \ldots$ of elements of P. Indeed, let us choose $x_0 \in E \cap (\cup\mathcal{I})$ and I_0 such that $I_0 \subset\downarrow x_0$. Suppose $x_0, \ldots, x_n$ and $I_0, \ldots, I_n$ defined such that $I_i \subset\downarrow x_i$ for all $i = 0, \ldots, n$. As $E \cap I_n \neq \emptyset$ we can select $x_n \in E \cap I_n$ and by definition of E, we can select some $I_{n+1} \in \mathcal{I}$ such that $I_{n+1} \subset\downarrow x_{n+1}$. Thus $\omega^* \leq P$.
Case (ii). There is some $I \in \mathcal{I}$ such that $I \cap E = \emptyset$. In particular all members of $\mathcal{I}$ included in I are unbounded in I. Since all infinite subchains of $\mathcal{I}$ are non-separating then, with no loss of generality, we may suppose that $I = A$ (hence $E = \emptyset$). We set $I_{-1} := A$ and define a sequence $x_0, I_0, \ldots, x_n, I_n, \ldots$ such that $I_n \in \mathcal{I}$, $x_n \in I_{n-1} \setminus I_n$ and $I_n \subseteq \{x_n\} \bigvee I$ for all $I \in \mathcal{I}$, all $n < \omega$. Members of this sequence being defined for all $n', n' < n$, observe that the set $\mathcal{I}_n := \{I \in \mathcal{I} : I \subseteq I_{n-1}\}$ being infinite is non-separating, hence there are $I \in \mathcal{I}_n$ and $x \in I_{n-1} \setminus I$ such that $I \subseteq \{x\} \bigvee J$ for all $J \in \mathcal{I}_n$. Set $I_n := I$ and $x_n := x$. Next, we define a sequence $y_0 := x_0, \ldots, y_n, \ldots$ such that for every $n \geq 1$:
$a_n)$ $x_n \leq y_n \in I_{n-1}$;
$b_n)$ $y_n \not\leq y_0 \vee y_{n-1}$;
$c_n)$ $y_j \leq y_i \vee y_n$ for every $i \leq j \leq n$.

Suppose $y_0, \dots, y_{n-1}$ defined for some n, $n \geq 1$. Since I_{n-1} is unbounded, we may select $z \in I_{n-1}$ such that $z \not\leq y_0 \vee \dots \vee y_{n-1}$. If $n = 1$, we set $y_1 := x_1 \vee z$. Suppose $n \geq 2$. Let $0 \leq j \leq n-2$. Since $y_{j+1} \vee \dots \vee y_{n-1} \in I_j \subseteq \{x_j\} \bigvee I_{n-1}$ we may select $t_j \in I_{n-1}$ such that $y_{j+1} \vee \dots \vee y_{n-1} \leq x_j \vee t_j$. Set $t := t_0 \vee \dots \vee t_{n-2}$ and $y_n := x_n \vee z \vee t$.
Let $f : \Omega(\omega^*) \to P$ be defined by $f(i,j) := y_i \vee y_j$ for all (i,j), $i < j < \omega$.
Condition $c_n)$ insures that f is join-preserving. Indeed, let $(i,j), (i',j') \in \Omega(\omega^*)$. We have $(i,j) \vee (i',j') = (i \wedge i', j \vee j')$ hence $f((i,j) \vee (i',j')) = f(i \wedge i', j \vee j') = y_{i \wedge i'} \vee y_{j \vee j'}$. If F is a finite subset of ω with minimum a and maximum b then conditions $c_n)$ force $\bigvee\{y_n : n \in F\} = y_a \vee y_b$. If $F := \{i, j, i', j'\}$ then, taking account of $i < j$ and $i' < j'$, we have $f(i,j) \vee f(i',j') = y_i \vee y_j \vee y_{i'} \vee y_{j'} = y_{i \wedge i'} \vee y_{j \vee j'}$. Hence $f((i,j) \vee (i',j')) = f(i,j) \vee f(i',j')$, proving our claim.
Next, f is one-to-one. Let $(i,j), (i',j') \in \Omega(\omega^*)$ such that $f(i,j) = f(i',j')$, that is $y_i \vee y_j = y_{i'} \vee y_{j'}$ (1). Suppose $j < j'$. Since $0 \leq i < j$, Condition $c_j)$ implies $y_i \leq y_0 \vee y_j$. In the other hand, since $0 \leq j \leq j'-1$, Condition $c_{j'-1})$ implies $y_j \leq y_0 \vee y_{j'-1}$. Hence $y_i \vee y_j \leq y_0 \vee y_{j'-1}$. From (1) we get $y_{j'} \leq y_0 \vee y_{j'-1}$, contradicting Condition $b_{j'})$. Hence $j' \leq j$. Exchanging the roles of j, j' gives $j' \leq j$ thus $j = j'$. If $i < i'$ then, Conditions $a_{i'})$ and $a_{j'})$ assure $y_{i'} \in I_{i'-1}$ and $y_{j'} \in I_{j'-1}$. Since $I_{j'-1} \subseteq I_{i'-1}$ we have $y_{i'} \vee y_{j'} \in I_{i'-1}$. In the other hand $x_i \notin I_i$ and $x_i \leq y_i \vee y_j$ thus $y_i \vee y_j \notin I_i$. From $I_{i'-1} \subseteq I_i$, we have $y_i \vee y_j \notin I_{i'-1}$, hence $y_i \vee y_j \neq y_{i'} \vee y_{j'}$ and $i' \leq i$. Similarly we get $i \leq i'$. Consequently $i = i'$. □

3.1 Proof of Proposition 1.2

If $\underline{\Omega}(\omega^*)$ embeds in $[\omega]^{<\omega}$ then $[\omega]^{<\omega}$ contains a non-separating ω^*-chain of ideals. This is impossible: a non-separating chain of ideals of $[\omega]^{<\omega}$ has necessarily a least element. Indeed, if the pair x, I ($x \in [\omega]^{<\omega}$, $I \in \mathcal{I}$) witnesses the fact that the chain $\mathcal{I}$ is non-separating then there are at most $\mid x \mid +1$ ideals belonging to $\mathcal{I}$ which are included in I (note that the set $\{\cup I \setminus \cup J : J \subseteq I, J \in \mathcal{I}\}$ is a chain of subsets of x). The proof of the general case requires more care. If $\underline{\Omega}(\omega^*)$ embeds in $I_{<\omega}(Q)$ as a join-semilattice then we may find a sequence $x_0, I_0, \dots, x_n, I_n, \dots$ such that $I_n \subset I_{n-1} \in J(I_{<\omega}(Q))$, $x_n \in I_{n-1} \setminus I_n$ and $I_n \subseteq \{x_n\} \bigvee I_m$ for every $n < \omega$ and every $m < \omega$. Set $I_\omega := \bigcap\{I_n : n < \omega\}$, $\overline{I}_n := \cup I_n$ for every $n \leq \omega$, $Q' := Q \setminus \overline{I}_\omega$ and $y_n := x_n \setminus \overline{I}_\omega$ for every $n < \omega$. We claim that $y_0, \dots, y_n, \dots$ form a strictly descending sequence in $I_{<\omega}(Q')$. According to Property $a)$ stated in Theorem 2.1, Q', thus Q, is not well-founded.
First, $y_n \in I_{<\omega}(Q')$. Indeed, if $a_n \in [Q]^{<\omega}$ generates $x_n \in I_{<\omega}(Q)$ then,

since $\overline{I}_\omega \in I(Q)$, $a_n \setminus \overline{I}_\omega$ generates $x_n \setminus \overline{I}_\omega \in I(Q')$. Next, $y_{n+1} \subset y_n$. It suffices to prove that the following inclusions hold:

$$x_{n+1} \cup \overline{I}_\omega \subseteq \overline{I}_n \subset x_n \cup \overline{I}_\omega$$

Indeed, substracting $\overline{I}_\omega$, from the sets figuring above, we get:

$$y_{n+1} = (x_{n+1} \cup \overline{I}_\omega) \setminus \overline{I}_\omega \subset (x_n \cup \overline{I}_\omega) \setminus \overline{I}_\omega = y_n$$

The first inclusion is obvious. For the second note that, since $J(I_{<\omega}(Q))$ is isomorphic to $I(Q)$, complete distributivity holds, hence with the hypotheses on the sequence $x_0, I_0, \ldots, x_n, I_n, \ldots$ we have $I_n \subseteq \bigcap\{\{x_n\} \bigvee I_m : m < \omega\} = \{x_n\} \bigvee \bigcap\{I_m : m < \omega\} = \{x_n\} \bigvee I_\omega$, thus $\overline{I}_n \subset x_n \cup \overline{I}_\omega$. □

Remark 3.3 *One can deduce the fact that $\Omega(\omega^*)$ does not embed as a join-semilattice in $[\omega]^{<\omega}$ from the fact that it contains a strictly descending chain of completely meet-irreducible ideals (namely the chain $\mathcal{I} := \{I_n : n < \omega\}$ where $I_n := \{(i, j) : n \leq i < j < \omega\}$) (see Proposition 4.10) but this fact by itself does not prevent the existence of some well-founded poset Q such that $\Omega(\omega^*)$ embeds as a join semilattice in $I_{<\omega}(Q)$.*

3.2 Proof of Theorem 1.3

In terms of join-semilattices and ideals, result becomes this: let P be a join-semilattice, then $J(P)$ is well-founded if and only if P is well-founded and contains no join-subsemilattice isomorphic to $\Omega(\omega^*)$ or to $[\omega]^{<\omega}$.

The proof goes as follows. Suppose that $J(P)$ is not well-founded. If some ω^*-chain in $J(P)$ is separating then, according to Lemma 3.1, P contains an infinite independent set. From Theorem 1.4, it contains a join-subsemilattice isomorphic to $[\omega]^{<\omega}$. If no ω^*-chain in $J(P)$ is separating, then all the infinite subchains of an arbitrary ω^*-chain are non-separating. From Lemma 3.2, either ω^* or $\Omega(\omega^*)$ embed in P as a join-semilattice. The converse is obvious. □

4 Join-subsemilattices of $I_{<\omega}(Q)$ and proof of Theorem 1.6

In this section, we consider join-semilattices which embed in join-semilattices of the form $I_{<\omega}(Q)$. These are easy to characterize internally (see Proposi-

tion 4.4). This is also the case if the posets Q are antichains (see Proposition 4.10) but does not go so well if the posets Q are well-founded (see Lemma 4.8).

Let us recall that if P is a join-semilattice, an element $x \in P$ is *join-prime* (or prime if there is no confusion), if it is distinct from the least element 0, if any, and if $x \leq a \vee b$ implies $x \leq a$ or $x \leq b$. This amounts to the fact that $P \setminus \uparrow x$ is an ideal . We denote $\mathbb{J}_{pri}(P)$, the set of join-prime members of P. We recall that $\mathbb{J}_{pri}(P) \subseteq \mathbb{J}_{irr}(P)$; the equality holds provided that P is a distributive lattice. It also holds if $P = I_{<\omega}(Q)$. Indeed:

Fact 4.1 *For an arbitrary poset Q, we have:*

$$\mathbb{J}_{irr}(I_{<\omega}(Q)) = \mathbb{J}_{pri}(I_{<\omega}(Q)) = down(Q) \tag{1}$$

Fact 4.2 *For a poset P, the following properties are equivalent:*

- *P is isomorphic to $I_{<\omega}(Q)$ for some poset Q;*
- *P is a join-semilattice with a least element in which every element is a finite join of primes.*

Proof. Observe that the primes in $I_{<\omega}(Q)$, are the $\downarrow x$, $x \in Q$. Let $I \in I_{<\omega}(Q)$ and $F \in [Q]^{<\omega}$ generating I, we have $I = \cup\{\downarrow x : x \in F\}$. Conversely, let P be a join-semilattice with a 0. If every element in P is a finite join of primes, then $P \cong I_{<\omega}(Q)$ where $Q := \mathbb{J}_{pri}(P)$. □

Let L be a complete lattice . For $x \in L$, set $x^+ := \bigwedge\{y \in L : x < y\}$. We recall that $x \in L$ is *completely meet-irreducible* if $x = \bigwedge X$ implies $x \in X$, or -equivalently- $x \neq x^+$. We denote $\triangle(L)$ the set of completely meet-irreducible members of L. We recall the following Lemma.

Lemma 4.3 *Let P be a join-semilattice, $I \in J(P)$ and $x \in P$. Then $x \in I^+ \setminus I$ if and only if I is a maximal ideal of $P \setminus \uparrow x$.*

Proposition 4.4 *Let P be a join-semilattice. The following properties are equivalent:*

(i) P embeds in $I_{<\omega}(Q)$, as a join-semilattice, for some poset Q;

(ii) P embeds in $I_{<\omega}(J(P))$ as a join-semilattice;

(iii) P embeds in $I_{<\omega}(\triangle(J(P)))$ as a join-semilattice;

(iv) For every $x \in P$, $P \setminus \uparrow x$ is a finite union of ideals.

Proof. $(i) \Rightarrow (iv)$ Let φ be an embedding from P in $P' := I_{<\omega}(Q)$. We may suppose that P has a least element 0 and that $\varphi(0) = \emptyset$ (if P has no least element, add one, say 0, and set $\varphi(0) := \emptyset$; if P has a least element, say a, and $\varphi(a) \neq \emptyset$, add to P an element 0 below a and set $\varphi(0) := \emptyset$). For $J' \in \mathfrak{P}(P')$, let $\varphi^{-1}(J') := \{x \in P : \varphi(x) \in J'\}$. Since φ is order-preserving, $\varphi^{-1}(J') \in I(P)$ whenever $J' \in I(P')$; moreover, since φ is join-preserving, $\varphi^{-1}(J') \in J(P)$ whenever $J' \in J(P')$. Now, let $x \in P$. We have $\varphi^{-1}(P' \setminus \varphi(x)) := P \setminus \uparrow x$. Since $\varphi(x)$ is a finite join of primes, $P' \setminus \uparrow \varphi(x)$ is a finite union of ideals. Since their inverse images are ideals, $P \setminus \uparrow x$ is a finite union of ideals too.

$(iv) \Rightarrow (iii)$ We use the well-known method for representing a poset by a family of sets.

Fact 4.5 *Let P be a poset and $Q \subseteq I(P)$. For $x \in P$ set $\varphi_Q(x) := \{J \in Q : x \notin J\}$. Then:*

(a) $\varphi_Q(x) \in I(Q)$;

(b) $\varphi_Q : P \to I(Q)$ is an order-preserving map;

(c) φ_Q is an order-embedding if and only if for every $x, y \in P$ such that $x \not\leq y$ there is some $J \in Q$ such that $x \notin J$ and $y \in J$.

Applying this to $Q := \triangle(J(P))$ we get immediately that φ_Q is join-preserving . Moreover, $\varphi_Q(x) \in I_{<\omega}(Q)$ if and only if $P \setminus \uparrow x$ is a finite union of ideals. Indeed, we have $P \setminus \uparrow x = \cup \varphi_Q(x)$, proving that $P \setminus \uparrow x$ is a finite union of ideals provided that $\varphi_Q(x) \in I_{<\omega}(Q)$. Conversely, if $P \setminus \uparrow x$ is a finite union of ideals, say $I_0, \ldots, I_n$, then since ideals are prime members of $I(P)$, every ideal included in I is included in some I_i, proving that $\varphi_Q(x) \in I_{<\omega}(Q)$. To conclude, note that if P is a join-semilattice then φ_Q is join-preserving.

$(iii) \Rightarrow (ii)$ Trivial.

$(ii) \Rightarrow (i)$ Trivial.

□

Corollary 4.6 *If a join-semilattice P has no infinite antichain, it embeds in $I_{<\omega}(J(P))$ as a join-subsemilattice.*

Proof. As is well known, if a poset has no infinite antichain then every initial segment is a finite union of ideals (cf [7], see also [8] 4.7.3 pp. 125). Thus Proposition 4.4 applies. □

Another corollary of Proposition 4.4 is the following.

Corollary 4.7 *Let P be a join-semilattice. If for every $x \in P$, $P \setminus \uparrow x$ is a finite union of ideals and $\triangle(J(P))$ is well-founded then P embeds as a join-subsemilattice in $I_{<\omega}(Q)$, for some well-founded poset Q.*

The converse does not hold:

Example 4.8 *There is a bipartite poset Q such that $I_{<\omega}(Q)$ contains a join-semilattice P for which $\triangle(J(P))$ is not well-founded.*

Proof. Let $\underline{2} := \{0, 1\}$ and $Q := \mathbb{N} \times \underline{2}$. Order Q in such a way that $(m, i) < (n, j)$ if $m > n$ in $\mathbb{N}$ and $i < j$ in $\underline{2}$.

Let P be the set of subsets X of Q of the form $X := F \times \{0\} \cup G \times \{1\}$ such that F is a non-empty final segment of $\mathbb{N}$, G is a non-empty finite subset of $\mathbb{N}$ and

$$min(F) - 1 \leq min(G) \leq min(F) \tag{2}$$

where $min(F)$ and $min(G)$ denote the least element of F and G w.r.t. the natural order on $\mathbb{N}$. For each $n \in \mathbb{N}$, let $I_n := \{X \in P : (n, 0) \notin X\}$.

Claim

1. Q is bipartite and P is a join-subsemilattice of $I_{<\omega}(Q)$.

2. The I_n's form a strictly descending sequence of members of $\triangle(J(P))$.

Proof of the Claim

1. The poset Q is decomposed into two antichains, namely $\mathbb{N} \times \{0\}$ and $\mathbb{N} \times \{1\}$ and for this raison is called *bipartite*. Next, P is a subset of $I_{<\omega}(Q)$. Indeed, Let $X \in P$. Let F, G such that $X = F \times \{0\} \cup G \times \{1\}$. Set $G' := G \times \{1\}$. If $min(G) = min(F) - 1$, then $X = \downarrow G'$ whereas if $min(G) = min(F)$ then $X = \downarrow G' \cup \{(min(F), 0)\}$. In both cases $X \in I_{<\omega}(Q)$. Finally, P is a join-semilattice. Indeed, let $X, X' \in P$ with $X := F \times \{0\} \cup G \times \{1\}$ and $X' := F' \times \{0\} \cup G' \times \{1\}$. Obviously $X \cup X' = (F \cup F') \times \{0\} \cup (G \cup G') \times \{1\}$. Since $X, X' \in P$, $F \cup F'$ is a non-empty final segment of $\mathbb{N}$ and $G \cup G'$ is a non-empty finite subset of $\mathbb{N}$. We have $min(G \cup G') = min(\{min(G), min(G')\}) \leq min(\{min(F), min(F')\}) = min(F \cup F')$ and similarly $min(F \cup F') - 1 = min\{min(F), min(F')\} - 1 = min\{min(F) - 1, min(F') - 1\} \leq min\{min(G), min(G')\} = min(G \cup G')$, proving that inequalities as in (2) hold. Thus $X \cup X' \in I_{<\omega}(Q)$.

2. Due to its definition, I_n is an non-empty initial segment of P which is closed under finite unions, hence $I_n \in J(P)$. Let $X_n := \{(n, 1), (m, 0) : m \geq n + 1\}$ and $Y_n := X_n \cup \{(n, 0)\}$. Clearly, $X_n \in I_n$ and $Y_n \in P$. We claim that $I_n^+ = I_n \bigvee \{Y_n\}$. Indeed, let J be an ideal containing strictly I_n.

Let $Y := \{m \in \mathbb{N} : m \geq p\} \times \{0\} \cup G \times \{1\} \in J \setminus I_n$. Since $Y \notin I_n$, we have $p \leq n$ hence $Y_n \subseteq Y \cup X_n \in J$. It follows that $Y_n \in J$, thus $I_n^+ \subseteq J$, proving our claim. Since $I_n^+ \neq I_n$, $I_n \in \triangle(J(P))$. Since, trivially, $I_n^+ \subseteq I_{n-1}$ we have $I_n \subset I_{n-1}$, proving that the I_n's form a strictly descending sequence. $\square$

Let E be a set and $\mathcal{F}$ be a subset of $\mathfrak{P}(E)$, the power set of E. For $x \in E$, set $\mathcal{F}_{\neg x} := \{F \in \mathcal{F} : x \notin F\}$ and for $X \subset \mathcal{F}$, set $\overline{X} := \bigcup X$. Let $\mathcal{F}^{<\omega}$ (resp. $\mathcal{F}^{\cup}$) be the collection of finite (resp. arbitrary) unions of members of $\mathcal{F}$. Ordered by inclusion, $\mathcal{F}^{\cup}$ is a complete lattice , the least element and the largest element being the empty set and $\bigcup \mathcal{F}$, respectively.

Lemma 4.9 *Let Q be a poset, $\mathcal{F}$ be a subset of $I_{<\omega}(Q)$ and $P := \mathcal{F}^{<\omega}$ ordered by inclusion.*

(a) The map $X \to \overline{X}$ is an isomorphism from $J(P)$ onto $\mathcal{F}^{\cup}$ ordered by inclusion.

(b) If $I \in \triangle(J(P))$ then there is some $x \in Q$ such that $I = P_{\neg x}$.

(c) If $\downarrow q$ is finite for every $q \in Q$ then $\overline{I^+} \setminus \overline{I}$ is finite for every $I \in J(P)$ and the set $\varphi_{\triangle}(X) := \{I \in \triangle(J(P)) : X \notin I\}$ is finite for every $X \in P$.

Proof.
(a) Let I and J be two ideals of P. Then J contains I if and only if $\overline{J}$ contains $\overline{I}$. Indeed, if $I \subseteq J$ then, clearly $\overline{I} \subseteq \overline{J}$. Conversely, suppose $\overline{I} \subseteq \overline{J}$. If $X \in I$, then $X \subseteq \overline{I}$, thus $X \subseteq \overline{J}$. Since $X \in I_{<\omega}(Q)$, and $X \subseteq \overline{J}$, there are $X_1, \ldots, X_n \in J$ such that $X \subseteq Y = X_1 \cup \ldots \cup X_n$. Since J is an ideal $Y \in J$. It follows that $X \in J$.
(b) Let $I \in \triangle(J(P))$. From (a), we have $\overline{I} \subset \overline{I^+}$. Let $x \in \overline{I^+} \setminus \overline{I}$. Clearly $P_{\neg x}$ is an ideal containing I. Since $x \notin \overline{P_{\neg x}}$, $P_{\neg x}$ is distinct from I^+. Hence $P_{\neg x} = I$. Note that the converse of assertion (b) does not hold in general.
(c) Let $I \in \triangle(J(P))$ and $X \in I^+ \setminus I$. We have $\{X\} \bigvee I = I^+$, hence from(a) $\overline{\{X\} \bigvee I} = \overline{I^+}$. Since $\overline{\{X\} \bigvee I} = X \cup \overline{I}$ we have $\overline{I^+} \setminus \overline{I} \subseteq X$. From our hypothesis on P, X is finite, hence $\overline{I^+} \setminus \overline{I}$ is finite. Let $X \in P$. If $I \in \varphi_{\triangle}(X)$ then according to (b) there is some $x \in Q$ such that $I = P_{\neg x}$. Necessarily $x \in X$. Since X is finite, the number of these I's is finite. $\square$

Proposition 4.10 *Let P be a join-semilattice. The following properties are equivalent:*

(i) P embeds in $[E]^{<\omega}$ as a join-subsemilattice for some set E;

(ii) for every $x \in P$, $\varphi_{\triangle}(x)$ is finite.

Proof. $(i) \Rightarrow (ii)$ Let φ be an embedding from P in $[E]^{<\omega}$ which preserves joins. Set $\mathcal{F} := \varphi(P)$. Apply part (c) of Lemma 4.9 . $(ii) \Rightarrow (i)$ Set $E := \triangle(J(P))$. We have $\varphi_{\triangle}(x) \in [E]^{<\omega}$. According to Fact 4.5 and Lemma 4.3, the map $\varphi_{\triangle} : P \rightarrow [E]^{<\omega}$ is an embedding preserving joins. □

Corollary 4.11 *Let β be a countable order type. If a proper initial segment contains infinitely many non-principal initial segments then no sierpinskisation P of β with ω can embed in $[\omega]^{<\omega}$ as a join-semilattice (whereas it embeds as a poset).*

Proof. According to Proposition 4.10 it suffices to prove that P contains some x for which $\varphi_{\Delta}(x)$ is infinite.

Let P be a sierpinskisation of β and ω. It is obtained as the intersection of two linear orders L, L' on the same set and having respectively order type β and ω. We may suppose that the ground set is $\mathbb{N}$ and L' the natural order.

Claim 1 A non-empty subset I is a non-principal ideal of P if and only if this is a non-principal initial segment of L.

Proof of Claim 1 Suppose that I is a non-principal initial segment of L. Then, clearly, I is an initial segment of P. Let us check that I is up-directed. Let $x, y \in I$; since I is non-principal in L, the set $A := I \cap \uparrow_L x \cap \uparrow_L y$ of upper-bounds of x and y w.r.t. L which belong to I is infinite; since $B := \downarrow_{L'} x \cup \downarrow_{L'} y$ is finite, $A \setminus B$ is non-empty. An arbitrary element $z \in A \setminus B$ is an upper bound of x, y in I w.r.t. the poset P proving that I is up-directed. Since I is infinite, I cannot have a largest element in P, hence I is a non-principal ideal of P. Conversely, suppose that I is a non-principal ideal of P. Let us check that I is an initial segment of L. Let $x \leq_L y$ with $y \in I$. Since I non-principal in P, $A := \uparrow_P y \cap I$ is infinite; since $B := \downarrow_{L'} x \cup \downarrow_{L'} y$ is finite, $A \setminus B$ is non-empty. An arbitrary element of $A \setminus B$ is an upper bound of x and y in I w.r.t. P. It follows that $x \in I$. If I has a largest element w.r.t. L then such an element must be maximal in I w.r.t. P, and since I is an ideal, I is a principal ideal, a contradiction.

Claim 2 Let $x \in \mathbb{N}$. If there is a non-principal ideal of L which does not contain x, there is a maximal one, say I_x. If P is a join-semilattice, $I_x \in \Delta(P)$.

Proof of Claim 2 The first part follows from Zorn's Lemma. The second part follows from Claim 1 and Lemma 4.3.

Claim 3 If an initial segment I of β contains infinitely many non-principal initial segments then there is an infinite sequence $(x_n)_{n<\omega}$ of elements of I such that the I_{x_n}'s are all distinct.

Proof of Claim 3 With Ramsey's theorem obtain a sequence $(I_n)_{n<\omega}$ of non-principal initial segments which is either strictly increasing or strictly decreasing . Separate two successive members by some element x_n and apply the first part of Claim 2.

If we pick $x \in \mathbb{N} \setminus I$ then it follows from Claim 3 and the second part of Claim 2 that $\varphi_\Delta(x)$ is infinite. □

Example 4.12 *If α is a countably infinite order type distinct from ω, $\Omega(\alpha)$ is not embeddable in $[\omega]^{<\omega}$ as a join-semilattice.*

Indeed, $\Omega(\alpha)$ is a sierpinskisation of $\omega\alpha$ and ω. And if α is distinct from ω, α contains some element which majorizes infinitely many others. Thus $\beta := \omega\alpha$ satisfies the hypothesis of Corollary 4.11.

Note that on an other hand, for every ordinal $\alpha \leq \omega$, there are representatives of $\Omega(\alpha)$ which are embeddable in $[\omega]^{<\omega}$ as join-semilattices.

Theorem 4.13 *Let Q be a well-founded poset and let $\mathcal{F} \subseteq I_{<\omega}(Q)$. The following properties are equivalent:*

1) *$\mathcal{F}$ has no infinite antichain;*
2) *$\mathcal{F}^{<\omega}$ is wqo;*
3) *$J(\mathcal{F}^{<\omega})$ is topologically scattered;*
4) *$\mathcal{F}^{\cup}$ is order-scattered;*
5) *$\mathfrak{P}(\omega)$ does not embed in $\mathcal{F}^{\cup}$;*
6) *$[\omega]^{<\omega}$ does not embed in $\mathcal{F}^{<\omega}$;*
7) *$\mathcal{F}^{\cup}$ is well-founded.*

Proof. We prove the following chain of implications:

$$1) \Longrightarrow 2) \Longrightarrow 3) \Longrightarrow 4) \Longrightarrow 5) \Longrightarrow 6) \Longrightarrow 7) \Longrightarrow 1)$$

1) $\Longrightarrow$ 2). Since Q is well-founded then, as mentioned in *a*) of Theorem 2.1, $I_{<\omega}(Q)$ is well-founded. It follows first that $\mathcal{F}^{<\omega}$ is well-founded, hence from Property *c*) of Theorem 2.1, every member of $\mathcal{F}^{<\omega}$ is a finite join of join-irreducibles. Next, as a subset of $\mathcal{F}^{<\omega}$, $\mathcal{F}$ is well-founded, hence wqo according to our hypothesis. The set of join-irreducible members of $\mathcal{F}^{<\omega}$ is wqo as a subset of $\mathcal{F}$. From Property *d*) of Theorem 2.1, $\mathcal{F}^{<\omega}$ is wqo

2) $\implies$ 3). If $\mathcal{F}^{<\omega}$ is wqo then $I(\mathcal{F}^{<\omega})$ is well-founded (cf. Property (b) of Theorem 2.1). If follows that $I(\mathcal{F}^{<\omega})$ is topologically scattered (cf.[17]); hence all its subsets are topologically scattered, in particular $J(\mathcal{F}^{<\omega})$.
3) $\implies$ 4). Suppose that $\mathcal{F}^{\cup}$ is not ordered scatered. Let $f : \eta \to \mathcal{F}^{\cup}$ be an embedding. For $r \in \eta$ set $\check{f}(r) = \bigcup\{f(r') : r' < r\}$. Let $X := \{\check{f}(r) : r < \eta\}$. Clearly $X \subseteq \mathcal{F}^{\cup}$. Furthermore X contains no isolated point (Indeed, since $\check{f}(r) = \bigcup\{\check{f}(r') : r' < r\}$, $\check{f}(r)$ belongs to the topological closure of $\{\check{f}(r') : r' < r\}$). Hence $\mathcal{F}^{\cup}$ is not topologically scatered.
4) $\implies$ 5). Suppose that $\mathfrak{P}(\omega)$ embeds in $\mathcal{F}^{\cup}$. Since $\eta \leq \mathfrak{P}(\omega)$, we have $\eta \leq \mathcal{F}^{\cup}$.
5) $\implies$ 6). Suppose that $[\omega]^{<\omega}$ embeds in $\mathcal{F}^{<\omega}$, then $J([\omega]^{<\omega})$ embeds in $J(\mathcal{F}^{<\omega})$. Lemma 4.9 assures that $J(\mathcal{F}^{<\omega})$ is isomorphic to $\mathcal{F}^{\cup}$. In the other hand $J([\omega]^{<\omega})$ is isomorphic to $\mathfrak{P}(\omega)$. Hence $\mathfrak{P}(\omega)$ embeds in $\mathcal{F}^{\cup}$.
6) $\implies$ 7). Suppose $\mathcal{F}^{\cup}$ not well-founded. Since Q is well-founded, $a)$ of Theorem 2.1 assures $I_{<\omega}(Q)$ well-founded, but $\mathcal{F}^{<\omega} \subseteq I_{<\omega}(Q)$, hence $\mathcal{F}^{<\omega}$ is well-founded. Furthermore, since $I_{<\omega}(Q)$ is closed under finite unions, we have $\mathcal{F}^{<\omega} \subseteq I_{<\omega}(Q)$, Proposition 1.2 implies that $\underline{\Omega}(\omega^*)$ does not embed in $\mathcal{F}^{<\omega}$. From Theorem 1.3, we have $\mathcal{F}^{<\omega}$ not well-founded.
7) $\implies$ 1). Clearly, $\mathcal{F}$ is well-founded. If $F_0, \ldots, F_n \ldots$ is an infinite antichain of members of $\mathcal{F}$, define $f(i,j) : [\omega]^2 \to Q$, choosing $f(i,j)$ arbitrary in $Max(F_i) \setminus F_j$. Divide $[\omega]^3$ into $R_1 := \{(i,j,k) \in [\omega]^3 : f(i,j) = f(i,k)\}$ and $R_2 := [\omega]^3 \setminus R_1$. From Ramsey's theorem, cf. [18], there is some infinite subset X of ω such that $[X]^3$ is included in R_1 or in R_2. The inclusion in R_2 is impossible since $\{f(i,j) : j < \omega\}$, being included in $Max(F_i)$, is finite for every i. For each $i \in X$, set $G_i := \bigcup\{F_j : i \leq j \in X\}$. This defines an ω^*-chain in $\mathcal{F}^{\cup}$. □

Remark 4.14 *If $\mathcal{F}^{<\omega}$ is closed under finite intersections then equivalence between* (3) *and* (4) *follows from Mislove's Theorem mentioned in [17].*

Theorem 4.13 above was obtained by the second author and M.Sobrani in the special case where Q is an antichain [19, 22] .

Corollary 4.15 *If P is a join-subsemilattice of a join-semilattice of the form $[\omega]^{<\omega}$, or more generally of the form $I_{<\omega}(Q)$ where Q is some well-founded poset, then $J(P)$ is well-founded if and only if P has no infinite antichain.*

Remark. If, in Theorem 4.13 above, we suppose that $\mathcal{F}$ is well-founded instead of Q, all implications in the above chain hold, except 6) $\Rightarrow$ 7). A counterexample is provided by $Q := \omega \oplus \omega^*$, the direct sum of the chains ω and ω^*, and $\mathcal{F}$, the image of $\underline{\Omega}(\omega^*)$ via a natural embedding.

4.1 Proof of Theorem 1.6

$(i) \Rightarrow (ii)$ Suppose that (i) holds. Set $Q := J(P)$. Since P contains no infinite antichain, P embeds as a join-subsemilattice in $I_{<\omega}(Q)$ (Corollary 4.6). From $b)$ of Theorem 2.1 Q is well-founded. Since P has no infinite antichain, it has no infinite independent set.

$(ii) \Rightarrow (i)$ Suppose that (ii) holds. Since Q is well-founded, then from $a)$ of Theorem 2.1, $I_{<\omega}(Q)$ is well-founded. Since P embeds in $I_{<\omega}(Q)$, P is well-founded. From our hypothesis, P contains no infinite independent set. According to implication $(iii) \Rightarrow (i)$ of Theorem 1.4 , it does not embed $[\omega]^{<\omega}$. From implication 6) $\Rightarrow$ 1) of Theorem 4.13, it has no infinite antichain. □

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